Chapter 6

Introduction to Return and Risk

Road Map

Part A Introduction to Finance.

Part B Valuation of assets, given discount rates.

Part C Determination of risk-adjusted discount rates.

- Introduction to return and risk.
- Portfolio theory.
- CAPM and APT.

Part D Introduction to derivative securities.

Main Issues

- Defining Risk
- Estimating Return and Risk
- Risk and Return - A Historical Perspective

1 Asset Returns

Asset returns over a given period are often uncertain:

$$\tilde{r} = \frac{\tilde{D}_1 + \tilde{P}_1 - P_0}{P_0} = \frac{\tilde{D}_1 + \tilde{P}_1}{P_0} - 1$$

where

- $\tilde{\cdot}$ denotes an uncertain outcome (random variable)
- P_0 is the price at the beginning of period
- $\tilde{P}_1$ is the price at the end of period - uncertain
- $\tilde{D}_1$ is the dividend at the end of period - uncertain.

Return on an asset is a *random variable*, characterized by

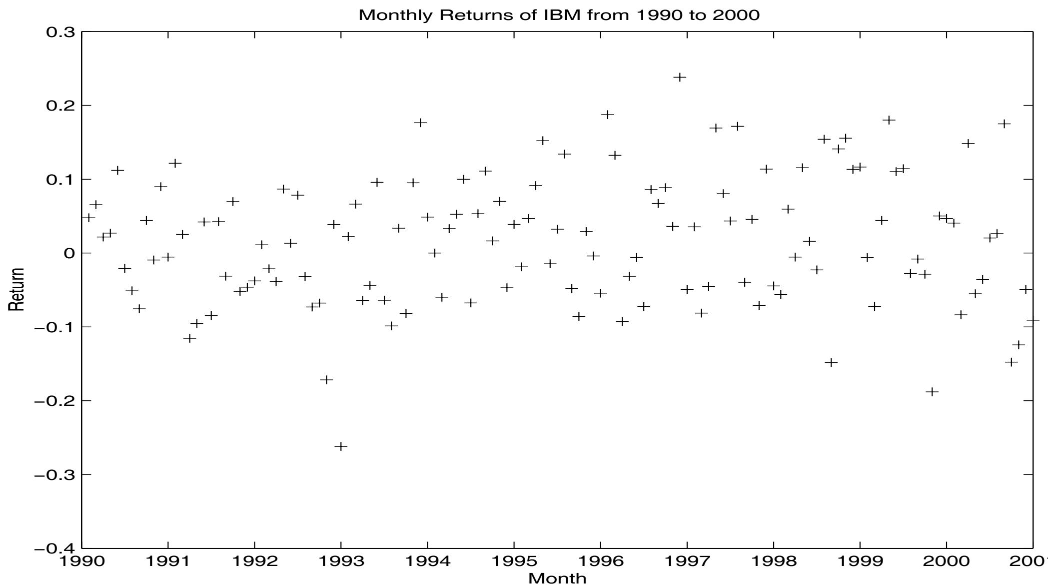
- all possible outcomes, and
- probability of each outcome (state).

Example. The S&P 500 index and the stock of MassAir, a regional airline company, give the following returns:

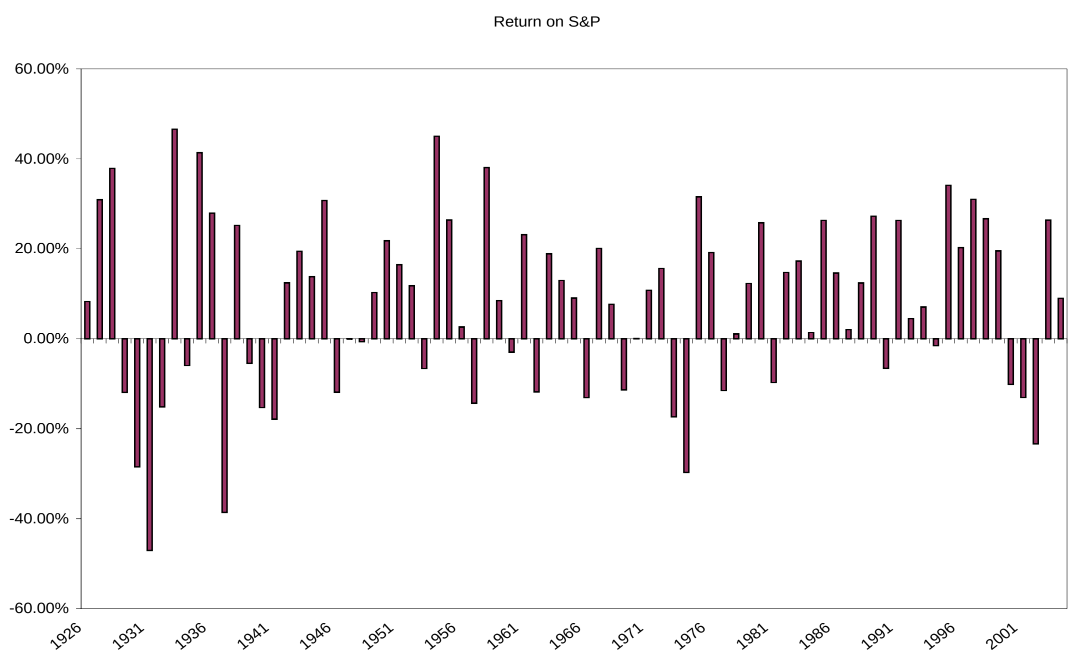
State	1	2	3
Probability	0.20	0.60	0.20
Return on S&P 500 (%)	- 5	10	20
Return on MassAir (%)	-10	10	40

Risk in asset returns can be substantial.

Monthly Returns - IBM (1990 – 2000)



Annual Returns - S&P 500 Index (1926 – 2004)



- Expected rate of return on an investment is the discount rate for its cash flows:

$$\bar{r} \equiv \mathrm{E}[\tilde{r}] = \frac{\mathrm{E}_0[\tilde{D}_1 + \tilde{P}_1]}{P_0} - 1$$

or

$$P_0 = \frac{\mathrm{E}_0[\tilde{D}_1 + \tilde{P}_1]}{1 + \bar{r}}$$

where $\bar{\cdot}$ denotes an expected value.

- Expected rate of return compensates for time-value and risk:

$$\bar{r} = r_F + \pi$$

where r_F is the risk-free rate and π is the risk premium

- r_F compensates for time-value
- π compensates for risk.

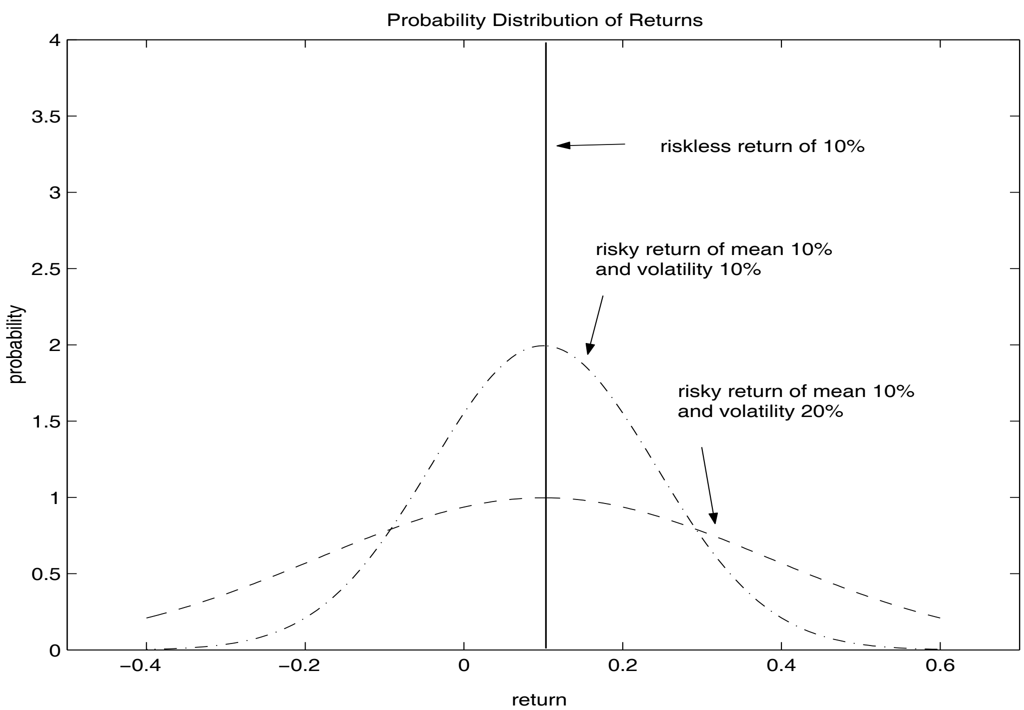
Questions:

1. How do we define and measure risk?
2. How are risks of different assets related to each other?
3. How is risk priced (how is π determined)?

2 Defining Risk

Example. Moments of return distribution. Consider three assets:

	Mean	StD
$\tilde{r}_0$ (%)	10.0	0.00
$\tilde{r}_1$ (%)	10.0	10.00
$\tilde{r}_2$ (%)	10.0	20.00



- Between Asset 0 and 1, which one would you choose?
- Between Asset 1 and 2, which one would you choose?

Investors care about expected return and risk.

Key Assumptions On Investor Preferences for 15.401

1. Higher mean in return is preferred:

   $$\bar{r} = \mathrm{E}[\tilde{r}].$$

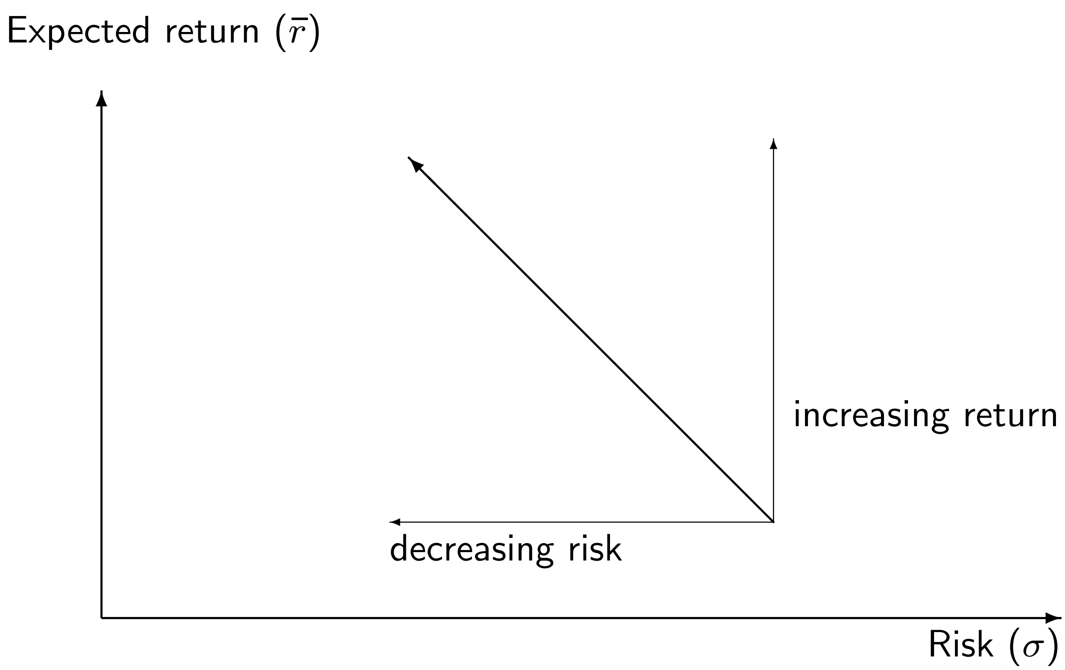
2. Higher standard deviation (StD) in return is disliked:

   $$\sigma = \sqrt{\mathrm{E}[(\tilde{r} - \bar{r})^2]}.$$

3. Investors care only about mean and StD (or variance).

Under 1-3, standard deviation (StD) gives a measure of risk.

Investor Preference for Return and Risk



3 Historical Return and Risk

Three central facts from history of U.S. financial markets:

1. Return on more risky assets has been higher on average than return on less risky assets:

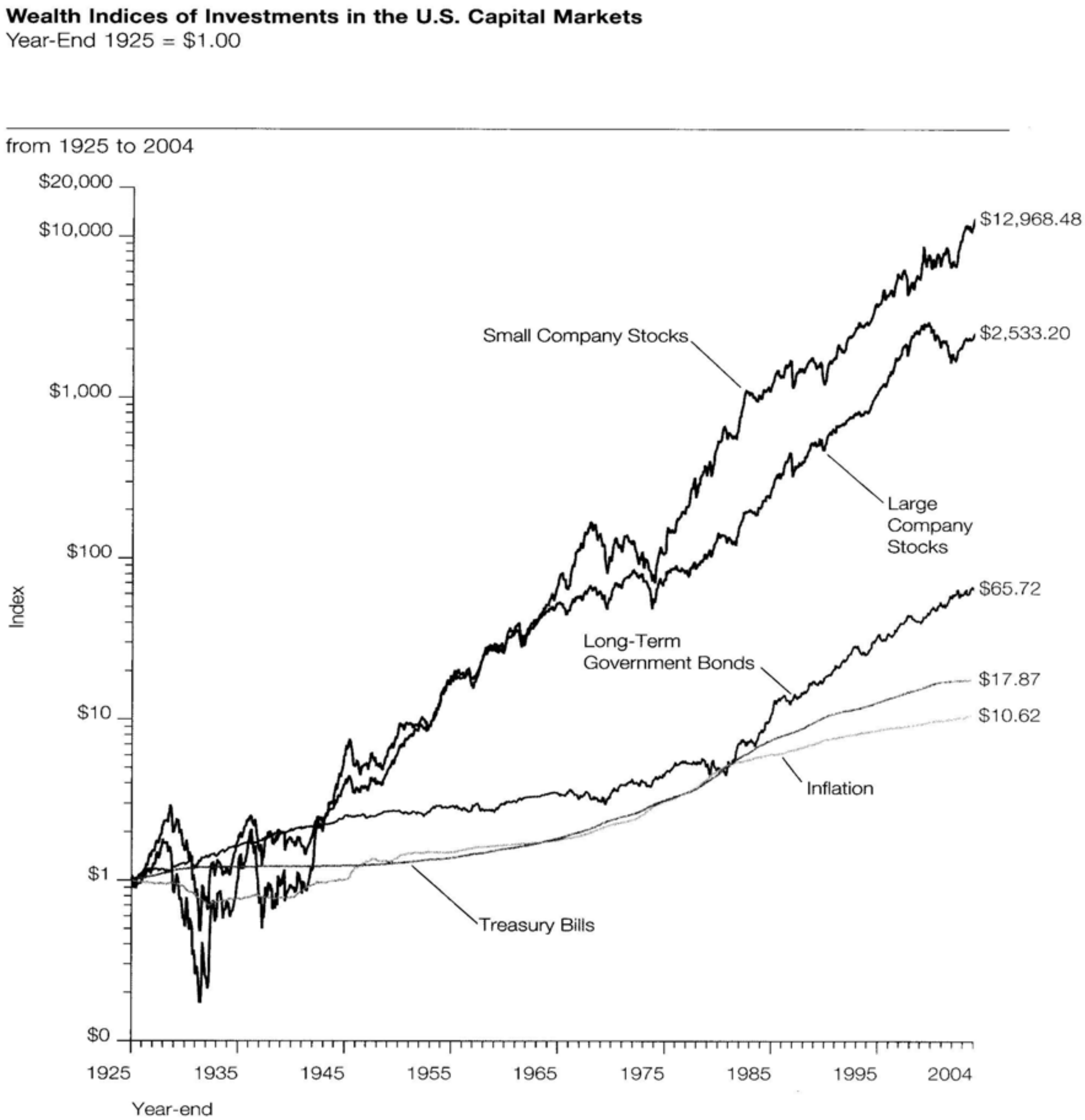
Average Annual Total Returns from 1926 to 2005 (Nominal)

Asset	Mean (%)	StD (%)
T-bills	3.8	3.1
Long term T-bonds	5.8	9.2
Long term corp. bonds	6.2	8.5
Large stocks	12.3	20.2
Small stocks	17.4	32.9
Inflation	3.1	4.3

Average Annual Total Returns from 1926 to 2005 (Real)

Asset	Mean (%)	StD (%)
T-bills	0.7	4.0
Long term T-bonds	2.9	10.4
Long term corp. bonds	3.2	9.7
Large stocks	9.1	20.3
Small stocks	13.9	32.3

Return Indices of Investments in the U.S. Capital Markets



Real returns from 1926 to 2004

Security	Initial	Total Return
T-Bills	$1.00	1.74
Long Term T-Bonds	$1.00	6.03
Corporate Bonds	$1.00	8.86
Large Stocks	$1.00	242.88
Small Stocks	$1.00	1,208.84

2. Returns on risky assets can be highly correlated to each other:

Cross Correlations of Annual Nominal Returns (1926 – 2005)

	Bills	T-bonds	C-bonds	L. stocks	S. stocks	Inflation
T-bills	1.00	0.23	0.20	-0.02	-0.10	0.41
L.T. T-bonds		1.00	0.93	0.12	-0.02	-0.14
L.t. C-bonds			1.00	0.19	0.08	-0.15
Large stocks				1.00	0.79	-0.02
Small stocks					1.00	0.04
Inflation						1.00

Cross Correlations of Annual Real Returns (1926 – 2005)

	Bills	T-bonds	C-bonds	L. stocks	S. stocks
T-bills	1.00	0.57	0.57	0.11	-0.06
L.T. T-bonds		1.00	0.95	0.20	0.02
L.t. C-bonds			1.00	0.26	0.11
Large stocks				1.00	0.79
Small stocks					1.00

3. Returns on risky assets are serially uncorrelated.

Serial Correlations of Annual Asset Returns (1926 – 2005)

	Serial Correlation	
Asset	Nominal return	Real return
T-bills ("risk-free")	0.91	0.67
Long term T-bonds	-0.08	0.02
Long term corp. bonds	0.08	0.19
Large stocks	0.03	0.02
Small stocks	0.06	0.03

(Note: The main source for the data in this subsection is *Stocks, bonds, bills and inflation, 2006 Year Book*, Ibbotson Associates, Chicago, 2006.)

4 Risk and Horizon

Previous discussions focused on return and risk over a fixed horizon. Often, we need to know:

- How do risk and return vary with horizon?
- How do risk and return change over time?

We need to know how successive asset returns are related.

IID Assumption: Asset returns are IID when successive returns are *independently and identically distributed*.

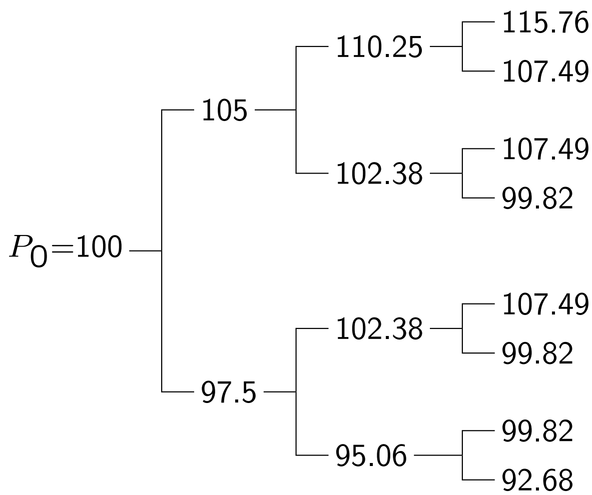
For IID returns, $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t$ are independent draws from the same distribution.

P_t is the asset price (including dividend). The continuously compounded return is

$$\frac{P_t}{P_{t-1}} = e^{\tilde{r}_t} \quad \text{or} \quad \log \frac{P_t}{P_{t-1}} = \log P_t - \log P_{t-1} = \tilde{r}_t.$$

Definition: Asset price (in log) follows a Random Walk when its changes are IID.

Example. An IID return series — a binomial tree for prices:



where

(1) price can go up by 5% or down by 2.5% at each node

(2) probabilities of "up" and "down" are the same at each node.

For the above binomial price process:

- Successive returns are independent and identically distributed.
- If "up" and "down" are equally likely, expected return is

$$(\log 1.05 + \log 0.975)/2 = 1.17\%.$$

- Return variance for one-period is

$$\sigma_1^2 = \left(\frac{1}{2}\log\frac{1.05}{0.975}\right)^2 = (0.0371)^2.$$

- Return variance over T periods is $(0.0371)^2 \times T$.

Implications of the IID assumption

(a) Returns are serially uncorrelated.

(b) No predicable trends, cycles or patterns in returns.

(c) Risk (measured by variance) accumulates linearly over time:

- Annual variance is 12 times the monthly variance.

Advantage of IID assumption:

- Future return distribution can be estimated from past returns.
- Return distribution over a given horizon provides sufficient information on returns for all horizons.
- IID assumption is consistent with information-efficient market.

Weakness of IID assumption:

- Return distributions may change over time.
- Returns may be serially correlated.
- Risk may not accumulate linearly over time.

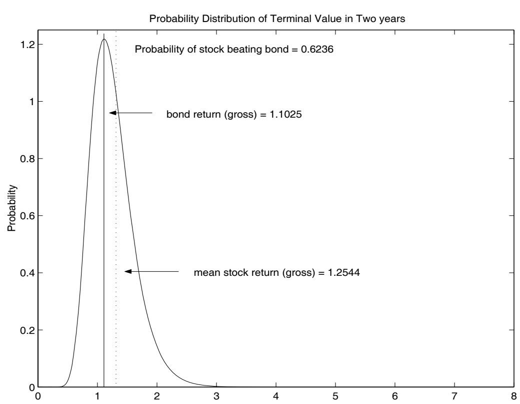
5 Investment in the long-run

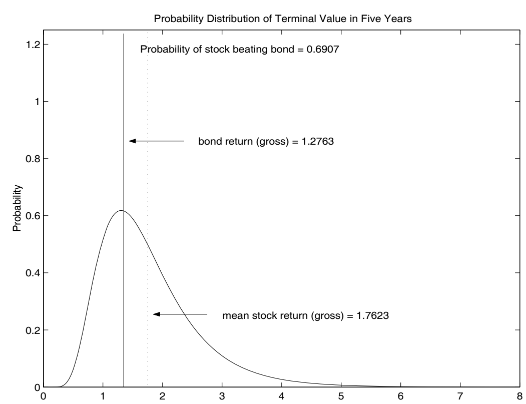
Are stocks less risky in the long-run? — Not if returns are IID.

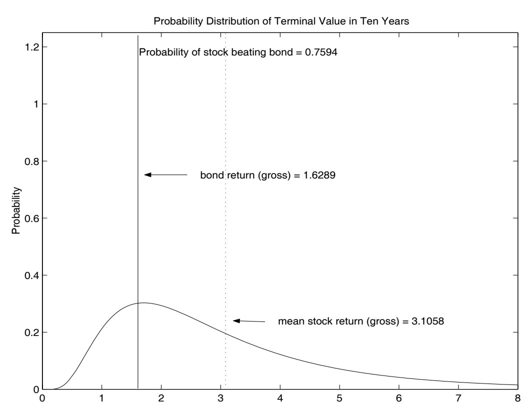
- Higher expected total return.
- Higher probability to outperform bond.
- More uncertainty about terminal value.

Example. Return profiles for different horizons.

- $r_{\text{bond}} = 5\%$.
- $r_{\text{stock}} = 12\%$ and $\sigma_{\text{stock}} = 20\%$.







6 Appendix: Probability and Statistics

Consider two random variables: $\tilde{x}$ and $\tilde{y}$

State	1	2	3	$\cdots$	n
Probability	p_1	p_2	p_3	$\cdots$	p_n
Value of $\tilde{x}$	x_1	x_2	x_3	$\cdots$	x_n
Value of $\tilde{y}$	y_1	y_2	y_3	$\cdots$	y_n

where $\sum_{i=1}^{n} p_i = 1$.

1. Mean: The expected or forecasted value of a random outcome.

$$\mathrm{E}[\tilde{x}] = \bar{x} = \sum_{j=1}^{n} p_j \cdot x_j.$$

2. Variance: A measure of how much the realized outcome is likely to differ from the expected outcome.

$$\mathsf{Var}[\tilde{x}] = \sigma_x^2 = \mathrm{E}\left[(\tilde{x} - \bar{x})^2\right] = \sum_{j=1}^{n} p_j \cdot \left(x_j - \bar{x}\right)^2.$$

Standard Deviation (Volatility):

$$\mathsf{StD}[\tilde{x}] = \sigma_x = \sqrt{\mathsf{Var}[\tilde{x}]}.$$

3. Skewness: A measure of asymmetry in distribtion.

$$\mathsf{Skew}[\tilde{x}] = \sqrt[3]{\mathrm{E}\left[(x - \bar{x})^3\right]}/\sigma_x.$$

4. Kurtosis: A measure of fatness in tail distribution.

$$\mathsf{Kurtosis}[\tilde{x}] = \sqrt[4]{\mathrm{E}\left[(x - \bar{x})^4\right]}/\sigma_x.$$

Example 1. Suppose that random variables $\tilde{x}$ and $\tilde{y}$ are the returns on S&P 500 index and MassAir, respectively, and

State	1	2	3
Probability	0.20	0.60	0.20
Return on S&P 500 (%)	- 5	10	20
Return on MassAir (%)	-10	10	40

- Expected Value:

$$\bar{x} = (0.2)(-0.05) + (0.6)(0.10) + (0.2)(0.20) = 0.09$$

$$\bar{y} = 0.12$$

- Variance:

$$\begin{aligned}\sigma_x^2 &= (0.2)(-0.05-0.09)^2 + \\ &\quad (0.6)(0.10-0.09)^2 + \\ &\quad (0.2)(0.20-0.09)^2 \\ &= 0.0064\end{aligned}$$

$$\sigma_y^2 = 0.0256$$

- Standard Deviation (StD):

$$\sigma_x = \sqrt{0.0064} = 8.00\%$$

$$\sigma_y = 16.00\%.$$

Covariance and Correlation

1. Covariance: A measure of how much two random outcomes "vary together".

$$\text{Cov}[\tilde{x}, \tilde{y}] = \sigma_{xy} = \mathrm{E}\left[(\tilde{x} - \bar{x})(\tilde{y} - \bar{y})\right] = \sum_{j=1}^{n} p_j \cdot \left(x_j - \bar{x}\right)\left(y_j - \bar{y}\right).$$

2. Correlation: A standardized measure of covariation.

$$\text{Corr}[\tilde{x}, \tilde{y}] = \rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}.$$

Note:

(a) ρ_{xy} must lie between -1 and 1.

(b) The two random outcomes are

- Perfectly positively correlated if $\rho_{xy} = +1$
- Perfectly negatively correlated if $\rho_{xy} = -1$
- Uncorrelated if $\rho_{xy} = 0$.

(c) If one outcome is certain, then $\rho_{xy} = 0$.

Example 1. (Continued.) For the returns on S&P and MassAir:

State	1	2	3
Probability	0.20	0.60	0.20
Return on S&P 500 ($\tilde{x}$) (%)	- 5	10	20
Return on MassAir ($\tilde{y}$) (%)	-10	10	40

with mean and StD:

$$\bar{x} = 0.09, \quad \sigma_x = 0.08,$$

$$\bar{y} = 0.12, \quad \sigma_y = 0.16.$$

We have

- Covariance:

$$\begin{aligned} \sigma_{xy} &= (0.2)(-0.05-0.09)(-0.10-0.12) + \\ &\quad (0.6)(0.10-0.09)(0.10-0.12) + \\ &\quad (0.2)(0.20-0.09)(0.40-0.12) \\ &= 0.0122. \end{aligned}$$

- Correlation:

$$\rho_{xy} = \frac{0.0122}{(0.08)(0.16)} = 0.953125.$$

Computation Rules

Let a and b be two constants.

$$\begin{aligned}
\mathrm{E}[a\tilde{x}] &= a\,\mathrm{E}[\tilde{x}]. \\
\mathrm{E}[a\tilde{x}+b\tilde{y}] &= a\,\mathrm{E}[\tilde{x}] + b\,\mathrm{E}[\tilde{y}]. \\
\mathrm{E}[\tilde{x}\tilde{y}] &= \mathrm{E}[\tilde{x}] \cdot \mathrm{E}[\tilde{y}] + \mathsf{Cov}[\tilde{x}, \tilde{y}]. \\
\mathsf{Var}[a\tilde{x}] &= a^2\mathsf{Var}[\tilde{x}] = a^2\sigma_x^2. \\
\mathsf{Var}[a\tilde{x}+b\tilde{y}] &= a^2\sigma_x^2 + b^2\sigma_y^2 + 2(ab)\sigma_{xy}. \\
\mathsf{Cov}[\tilde{x}+\tilde{y}, \tilde{z}] &= \mathsf{Cov}[\tilde{x}, \tilde{z}] + \mathsf{Cov}[\tilde{y}, \tilde{z}]. \\
\mathsf{Cov}[a\tilde{x}, b\tilde{y}] &= (ab)\mathsf{Cov}[\tilde{x}, \tilde{y}] = (ab)\sigma_{xy}.
\end{aligned}$$

Linear Regression

Relation between two random variables $\tilde{y}$ and $\tilde{x}$:

$$\tilde{y} = \alpha + \beta\tilde{x} + \tilde{\epsilon}$$

where

$$\beta = \frac{\mathsf{Cov}[\tilde{y},\tilde{x}]}{\mathsf{Var}[\tilde{x}]} = \frac{\sigma_{yx}}{\sigma_x^2}$$

$$\alpha = \bar{y} - \beta\bar{x}$$

$$\mathsf{Cov}[\tilde{x},\tilde{\epsilon}] = 0.$$

- β gives the expected deviation of $\tilde{y}$ from $\bar{y}$ for a given deviation of $\tilde{x}$ from $\bar{x}$.
- $\tilde{\epsilon}$ has zero mean: $\mathrm{E}[\tilde{\epsilon}] = 0$.
- $\tilde{\epsilon}$ represents the part of y that is uncorrelated with x:

  $\mathsf{Cov}[\tilde{x},\tilde{\epsilon}] = 0.$

Furthermore:

$$\begin{aligned}\sigma_y^2 = \mathsf{Var}[\tilde{y}] &= \mathsf{Var}[\alpha+\beta\tilde{x}+\tilde{\epsilon}] \\ &= \beta^2\sigma_x^2 + \sigma_\epsilon^2\end{aligned}$$

$$\begin{aligned}\text{Total Variance} &= \text{Explained Variance} \\ &\quad + \text{ Unexplained Variance.}\end{aligned}$$

- Explained variance: $\beta^2\sigma_x^2$
- Unexplained variance: σ_ϵ^2.

What fraction of the total variance of $\tilde{y}$ is explained by $\tilde{x}$?

$$R^2 = \frac{\text{Explained Variance}}{\text{Total Variance}} = \frac{\beta^2\sigma_x^2}{\sigma_y^2} = \frac{\beta^2\sigma_x^2}{\beta^2\sigma_x^2 + \sigma_\epsilon^2}.$$

Example 1. (Continued.) In the above example: $\tilde{x}$ is the return on S&P 500 and $\tilde{y}$ is the return on MassAir.

$$\beta = \frac{0.0122}{0.08^2} = 1.9062 \quad \text{and} \quad \alpha = -0.0516.$$

State	1	2	3
Probability	0.20	0.60	0.20
Return on S&P 500 (%)	- 5.00	10.00	20.00
Return on MassAir (%)	-10.00	10.00	40.00
$\tilde{\epsilon} = \tilde{y} - (\alpha + \beta\tilde{x})$ (%)	4.69	- 3.90	7.03

Moreover,

$$\sigma_x^2 = 0.0064, \quad \sigma_y^2 = 0.0256, \quad \sigma_\epsilon^2 = 0.00234$$

and

$$R^2 = \frac{(1.9062)^2(.0064)}{(.0256)} = 0.9084$$

$$1 - R^2 = 0.0916.$$

7 Homework

Readings:

- BKM Chapter 5.2–5.4.
- BMA Chapter 7.1.