UNIT-II QUANTIFIERS

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

The domain of a predicate variable is the set of all values that may be substituted in place of the variable.

Let P(x) be the predicate "$x^2 > x$" with domain the set R of all real numbers

P(2), P(1/ 2), and P(−1/ 2), and indicate which of these statements are true and which are false.

- If P(x) is a predicate and x has **domain D**, the truth set of P(x) is the set of all elements of D that make P(x) true when they are substituted for x.

The truth set of P(x) is denoted $\{x \in D \mid P(x)\}$.

Let Q(n) be the predicate “n is a factor of 8.”

Find the truth set of Q(n) if

a. the domain of n is the set Z+ of all positive integers {1,2,4,8}

b. the domain of n is the set Z of all integers. {-1,-2,-4,1,2,4,8}

Quantifiers are words that refer to quantities such as “some” or “all” and tell for how many elements a given predicate is true.

The symbol ∀ denotes “for all” and is called the universal quantifier.

The symbol ∃ denotes "there exists” and is called the existential quantifier.

- Let Q(x) be a predicate and D the domain of x.
- A universal statement is a statement of the form “$\forall x \in D, Q(x)$.”
- It is defined to be true if, and only if, Q(x) is true for every x in D.
- It is defined to be false if, and only if, Q(x) is false for at least one x in D.
- A value for x for which Q(x) is false is called a c**ounterexample** to the universal statement.

1. Let D ={ 1,2,3,4,5}, and consider the statement $\forall x \in D, x^2 \geq x$. Show that this statement is true.
2. Consider the statement $\forall x \in R, x^2 \geq x$.

Find a counterexample to show that this statement is false.

- Let $Q(x)$ be a predicate and D the domain of x.
- An existential statement is a statement of the form "$\exists\, x \in D$ such that $Q(x)$."
- It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D.
- It is false if, and only if, $Q(x)$ is false for all x in D.

Consider the statement

$\exists\, m \in Z^{+}$ such that $m^2 = m$. Show that this statement is true.

Let $E = \{5,6,7,8\}$ and consider the statement $\exists m \in E$ such that $m^2 = m$. Show that this statement is false.

- Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol ∀or ∃. a. $\forall x \in R, x2 \geq 0$. b. $\forall x \in R, x^2 \geq -1$. c. $\exists m \in Z^+$ such that $m^2 = m$. Solution
- a. All real numbers have nonnegative squares. Or: Every real number has a nonnegative square. Or: Any real number has a nonnegative square. Or: The square of each real number is nonnegative.

- The negation of a statement of the form
- $\forall x$ *in D, Q(x)*
- is logically equivalent to a statement of the form
- $\exists x$ *in D such that* $\sim Q(x)$.
- Symbolically, $\sim(\forall x \in D, Q(x)) \equiv \exists x \in D$ *such that* $\sim Q(x)$.

- The negation of a universal statement ("all are") is logically equivalent to an
- existential statement ("some are not" or "there is at least one that is not").

- The negation of a statement of the form
- $\exists x$ *in D such that Q(x)*
- is logically equivalent to a statement of the form
- $\forall x$ *in D,* $\sim Q(x)$.
- Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x)$.

- **The negation of an existential statement ("some are") is logically**
- **equivalent to a universal statement ("none are" or "all are not").**

Consider a statement of the form: $\forall x \in D$, *if P(x) then Q(x).*

1. Its **contrapositive is the statement:** $\forall x \in D$, ***if ~Q(x) then ~P(x).***

2. Its **converse is the statement:** $\forall x \in D$, ***if Q(x) then P(x).***

3. Its **inverse is the statement:** $\forall x \in D$, ***if ~P(x) then ~Q(x).***

If a real number is greater than 2, then its square is greater than 4.

Contrapositive: $\forall x \in R$, *if* $x2 \leq 4$ *then* $x \leq 2$.

Or: If the square of a real number is less than or equal to 4,

then the number is less than or equal to 2.

Converse: $\forall x \in R$, *if* $x2 > 4$ *then* $x > 2$.

Or: If the square of a real number is greater than 4, then the

number is greater than 2.

Inverse: $\forall x \in R$, *if* $x \leq 2$ *then* $x2 \leq 4$.

Or: If a real number is less than or equal to 2, then the square

of the number is less than or equal to 4.

1 Write a formal negation for each of the following statements:

a. ∀ fish *x, x has gills.*

b. ∀ computers *c, c has a CPU.*

c. ∃ a movie *m such that m is over 6 hours long.*

d. ∃ a band *b such that b has won at least 10 Grammy* awards.

2. Write an informal negation for each of the following statements.

Be careful to avoid negations that are ambiguous.

a. All dogs are friendly.

b. All people are happy.

c. Some suspicions were substantiated.

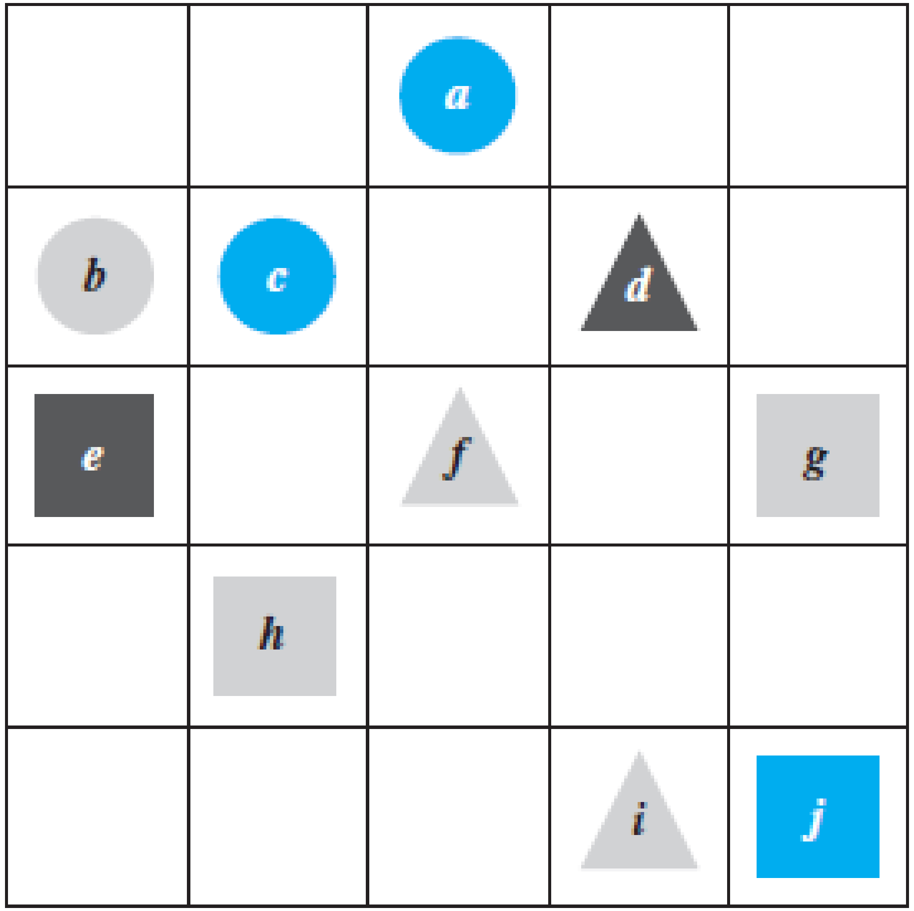
d. Some estimates are accurate.

5. Write a negation for each of the following statements.

a. Any valid argument has a true conclusion.

b. Every real number is positive, negative, or zero.

Consider the Tarski world shown in Figure 3.3.1.



- Let Triangle*(x), Circle(x), and Square(x) mean "x is a triangle," "x is a circle," and "x* is a square";

  let Blue*(x), Gray(x), and Black(x) mean "x is blue," "x is gray," and "x is* black";

  let RightOf*(x, y): "x is to the right* of *y,"*

  Above(x, y): "x is above y,"

  and SameColorAs(x, y) : "x has the same color as y";

  and use the notation x = y to denote the predicate "*x is equal to y".*

  Let the common domain D of all variables be the set of all the objects in the Tarski world.

- Use formal, logical notation to write each of the following statements, and write a formal negation for each statement.

a. For all circles *x, x is above f .*

b. There is a square *x such that x is black.*

c. For all circles *x, there is a square y such that x and y have the same color.*

d. There is a square *x such that for all triangles y, x is to right of y.*

- Solution
- a. *Statement:* $\forall x(\mathit{Circle}(x) \rightarrow \mathit{Above}(x, f))$.

Negation: $\sim(\forall x(\mathit{Circle}(x) \rightarrow \mathit{Above}(x, f)))$

$\equiv \exists x \sim (\mathit{Circle}(x) \rightarrow \mathit{Above}(x, f))$

by the law for negating a ∀ statement

$\equiv \exists x(\mathit{Circle}(x) \wedge \sim\mathit{Above}(x, f))$

by the law of negating an if-then statement

b. *Statement:* $\exists x(\mathit{Square}(x) \wedge \mathit{Black}(x))$.

Negation: $\sim(\exists x(\mathit{Square}(x) \wedge \mathit{Black}(x)))$

$\equiv \forall x \sim (\mathit{Square}(x) \wedge \mathit{Black}(x))$

by the law for negating a ∃ statement

$\equiv \forall x(\sim\mathit{Square}(x) \vee \sim\mathit{Black}(x))$

by De Morgan's law

c. *Statement:* $\forall x(\mathit{Circle}(x) \rightarrow \exists y(\mathit{Square}(y) \wedge \mathit{SameColor}(x, y)))$.

Negation: $\sim(\forall x(\mathit{Circle}(x) \rightarrow \exists y(\mathit{Square}(y) \wedge \mathit{SameColor}(x, y))))$

$\equiv \exists x \sim (\mathit{Circle}(x) \rightarrow \exists y(\mathit{Square}(y) \wedge \mathit{SameColor}(x, y)))$

by the law for negating a ∀ statement

$\equiv \exists x(\mathit{Circle}(x) \wedge \sim(\exists y(\mathit{Square}(y) \wedge \mathit{SameColor}(x, y))))$

by the law for negating an if-then statement

- $\equiv \exists x(Circle(x) \wedge \forall y(\sim(Square(y) \wedge SameColor(x, y))))$
- by the law for negating a ∃ statement
- $\equiv \exists x(Circle(x) \wedge \forall y(\sim Square(y) \vee \sim SameColor(x, y)))$
- by De Morgan’s law
- d. *Statement:* $\exists x(Square(x) \wedge \forall y(Triangle(y) \rightarrow RightOf(x, y)))$.
- *Negation:* $\sim(\exists x(Square(x) \wedge \forall y(Triangle(y) \rightarrow RightOf(x, y))))$
- $\equiv \forall x \sim (Square(x) \wedge \forall y(Triangle(x) \rightarrow RightOf(x, y)))$
- by the law for negating a ∃ statement
- $\equiv \forall x(\sim Square(x) \vee \sim(\forall y(Triangle(y) \rightarrow RightOf(x, y))))$
- by De Morgan’s law
- $\equiv \forall x(\sim Square(x) \vee \exists y(\sim(Triangle(y) \rightarrow RightOf(x, y))))$
- by the law for negating a ∀ statement
- $\equiv \forall x(\sim Square(x) \vee \exists y(Triangle(y) \wedge \sim RightOf(x, y)))$

- Write each of following statements informally and find its truth value.
- a. ∃ an item *I such that* ∀ *students S, S chose I .*
- b. ∃ a student *S such that* ∀ *items I, S chose I .*
- c. ∃ a student *S such that* ∀ *stations Z, ∃ an item I in Z such that S chose I .*
- d. ∀ students *S and* ∀ *stations Z, ∃ an item I in Z such that S chose I .*
- Solution
- a. There is an item that was chosen by every student. This is true; every student chose pie.
- b. There is a student who chose every available item. This is false; no student chose all
- nine items.
- c. There is a student who chose at least one item from every station. This is true; both
- Uta and Tim chose at least one item from every station.
- d. Every student chose at least one item from every station. This is false; Yuen did not
- choose a salad.